ON SYSTEMS WITH FRICTION

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Mechanical systems with friction constraints are usually reduced to systems with ideal constraints by supplementing the given forces by frictional forces and by adding certain relations obtained from empirical laws of friction. These laws can have various forms, depending on the nature of the constraints within the system, but they always have certain features in common. A general theory of motion of systems with friction was developed by Painlevé [1].

A direct application of Lagrange's method makes it possible to establish a general principle, even for systems with friction, that does not involve explicitly the constraint reactions. This so-called Euler-Lagrange principle was established by Appell [2] for displacements admissible by frictionless constraints which are orthogonal to the reactions of surfaces with friction. Chetaev [3] formulated this principle for admissible displacements that are orthogonal to the actual velocities of the points of the system.

Of special interest is the problem on the extension of another basic principle of mechanics, the principle of least constraint of Gauss, to systems with friction. The work [4] of Pozharitskii is devoted to this problem. By considering a system in which certain points are constrained to slide with Coulomb friction along given surfaces, and by assuming the knowledge of the normal components of the reactions of the latter, Pozharitskii proved that for the actual motion of such a system it is sufficient that a certain expression involving the forces of friction has to attain a minimum with respect to accelerations. It should be noted that the problems in which the normal reactions do not depend on friction must be considered as the more restricted ones and as the more simple ones [2]; in the more general cases, the normal reactions depend on the coefficient of sliding friction. Besides that, the experimental laws of friction can differ from Coulomb's law. There exist also large classes of systems with non-ideal constraints which fall into the general definition of systems with friction according to Painlevé [1].

It seems that the question on the extension of the principle of Gauss to general systems with friction has not yet been considered, even though the solution of this problem is of certain interest.

In this paper it is assumed that Painlevé's definition of systems with friction is applicable under any experimental laws of friction. An extension of Painlevé's results to nonholonomic systems is given. Furthermore, Gauss's principle is established for such systems in two forms: with explicit inclusion of the friction forces, and, what is more interesting, without the explicit involvement of the forces of friction. The equations of motion of the system with friction are derived from Gauss's principle.

1. Let us consider a system of n material points P_ν with masses m_ν, whose positions relative to a fixed coordinate system are given by the Cartesian coordinates

$$x_\nu,\ y_\nu,\ z_\nu \qquad (\nu = 1, \ldots, n)$$

Suppose that the given forces $F_\nu(X_\nu,\ Y_\nu,\ Z_\nu)$ act on the points P_ν, and that there exist certain geometric constraints

$$f_\alpha(x, y, z, t) = 0 \qquad (\alpha = 1, \ldots, p_1) \tag{1.1}$$

and also kinematic (in general, nonlinear) relations

$$\varphi_\beta(x, y, z, x', y', z', t) = 0 \qquad (\beta = 1, \ldots, p_2) \tag{1.2}$$

where x_ν', y_ν', z_ν' denote the projection vectors of the velocity of the point P_ν. The connections (1.1) and (1.2) are assumed to be independent of the given forces, and of each other.

Differentiating (1.1) with respect to time twice, and Equations (1.2) once, we obtain

$$\sum_\nu (a_{s\nu} x_\nu'' + b_{s\nu} y_\nu'' + c_{s\nu} z_\nu'') + e_s = 0 \qquad (s = 1, \ldots, p,\ p = p_1 + p_2) \tag{1.3}$$

where $a_{s\nu}$, $b_{s\nu}$, $c_{s\nu}$, e_s are known functions of the coordinates x_ν, y_ν, z_ν and the velocities x_ν', y_ν', z_ν' of the point P_ν ($\nu = 1, \ldots, n$), and of the time t.

The possible displacements $\delta r_\nu(\delta x_\nu,\ \delta y_\nu,\ \delta z_\nu)$ of the points P_ν are determined by p independent relations

$$\sum_\nu (a_{s\nu}\, \delta x_\nu + b_{s\nu}\, \delta y_\nu + c_{s\nu}\, \delta z_\nu) = 0 \qquad (s = 1, \ldots, p) \tag{1.4}$$

Thus, among the $3n$ changes of the coordinates of the points of the system there will be $k = 3n - p$ independent and p dependent ones. Equation (1.4) makes it possible to express the dependent variations in terms of the independent ones.

Suppose that by relations (1.1) the state of the system is determined by means of $l = 3n - p$ independent Lagrangean coordinates $q_1, \ldots, q_l$

$$x_\nu = x_\nu(q_1, \ldots, q_l, t), \qquad y_\nu = y_\nu(q_1, \ldots, q_l, t), \qquad z_\nu = z_\nu(q_1, \ldots, q_l, t) \qquad (\nu = 1, \ldots, n) \tag{1.5}$$

whose variations are, in view of the constraints (1.2), connected by p_2 equations

$$\sum_{j=1}^{l}\sum_{\nu}\left(a_{r\nu}\frac{\partial x_\nu}{\partial q_j} + b_{r\nu}\frac{\partial y_\nu}{\partial q_j} + c_{r\nu}\frac{\partial z_\nu}{\partial q_j}\right)\delta q_j = 0 \qquad (r = p_1 + 1, \ldots, p)$$

Expressing, by means of the last equations, the p_2 variations $\delta q_{l-p_2+1}, \ldots, \delta q_l$ in the form of linear homogeneous functions of the k remaining independent variations $\delta q_1, \ldots, \delta q_k$ and substituting the found expressions into the relations

$$\delta x_\nu = \sum_{j=1}^{l}\frac{\partial x_\nu}{\partial q_j}\delta q_j, \qquad \delta y_\nu = \sum_{j=1}^{l}\frac{\partial y_\nu}{\partial q_j}\delta q_j, \qquad \delta z_\nu = \sum_{j=1}^{l}\frac{\partial z_\nu}{\partial q_j}\delta q_j \quad (\nu = 1, \ldots, n) \tag{1.6}$$

we can express the latter in the form

$$\delta x_\nu = \sum_{i=1}^{k} A_{\nu i}\delta q_i, \qquad \delta y_\nu = \sum_{i=1}^{k} B_{\nu i}\delta q_i, \qquad \delta z_\nu = \sum_{i=1}^{k} C_{\nu i}\delta q_i \quad (\nu = 1, \ldots, n) \tag{1.7}$$

Here $A_{\nu i}$, $B_{\nu i}$, $C_{\nu i}$ are functions of the coordinates $q_1, \ldots, q_l$, of their derivatives $q_1', \ldots, q_l'$, and of time t; the variations $\delta q_1, \ldots, \delta q_k$ are arbitrary.

The constraints imposed on the system depend on the physical nature of the system. Hence the characteristic features of the constraints can be reduced to certain axioms which express the experimentally determined relations. In case of ideal constraints, such an axiom is given by the equation

$$\sum_{\nu}(R_{\nu x}\delta x_\nu + R_{\nu y}\delta y_\nu + R_{\nu z}\delta z_\nu) = 0$$

i.e. the sum of the elementary work of the constraint reactions $R_\nu(R_{\nu x}, R_{\nu y}, R_{\nu z})$ is equal to zero for every possible displacement of the system, whatever the position, the velocities, and the given forces F_ν may be at the given moment. If, however, the sum of the elementary work of the constraint reactions for all possible displacements of the system is not always equal to zero, then the given system is a system with friction [1].

Painlevé has studied holonomic system with friction, but many of his results can easily be extended to the case of nonholonomic systems with friction. We shall derive some of these extensions with their consequences.

It is easily seen that if the sum of the elementary work of a system of forces $F_{\nu x}$, $F_{\nu y}$, $F_{\nu z}$ for every possible displacement of a system is to be zero, then it is sufficient and necessary that the following equations hold:

$$F_{\nu x} = \sum_{s=1}^{p} \lambda_s a_{s\nu}, \quad F_{\nu y} = \sum_{s=1}^{p} \lambda_s b_{s\nu}, \quad F_{\nu z} = \sum_{s=1}^{p} \lambda_s c_{s\nu} \qquad (\nu = 1, \ldots, n) \quad (1.8)$$

where λ_s are coefficients that are the same for all points of the system. Let us prove this assertion.

Sufficiency. If Equations (1.8) hold, then

$$\sum_{\nu} (F_{\nu x}\,\delta x_\nu + F_{\nu y}\,\delta y_\nu + F_{\nu z}\,\delta z_\nu) = 0$$

in view of Equation (1.4).

Necessity. Suppose the last relation holds. Then for every possible displacement it is true that

$$\sum_{\nu} \left\{\left(F_{\nu x} - \sum_{s} \lambda_s a_{s\nu}\right)\delta x_\nu + \left(F_{\nu y} - \sum_{s} \lambda_s b_{s\nu}\right)\delta y_\nu + \left(F_{\nu z} - \sum_{s} \lambda_s c_{s\nu}\right)\delta z_\nu\right\} = 0$$

with still undetermined λ_s. By determining them in such a way that the coefficients of p dependent variations δx, δy, δz vanish in this equation, we obtain a sum which contains only k independent variations. Thus we obtain the $3n$ equations (1.8).

We next consider the constraint reactions $\mathbf{R}_\nu$ ($\nu = 1, \ldots, n$) whose sum of elementary work for all possible displacements is $\tau \neq 0$.

Obviously, there exists an infinite number of systems of forces $\mathbf{R}'(R'_{\nu x}, R'_{\nu y}, R'_{\nu z})$ which have the property that for all possible displacements

$$\sum (R'_{\nu x}\delta x_\nu + R'_{\nu y}\,\delta y_\nu + R'_{\nu z}\,\delta z_\nu) = \tau$$

In order that this may be true it is necessary and sufficient, on the basis of our proof, that

$$R'_{\nu x} = R_{\nu x} + \sum_{s} \lambda_s a_{s\nu}, \quad R'_{\nu y} = R_{\nu y} + \sum_{s} \lambda_s b_{s\nu}, \quad R'_{\nu z} = R_{\nu z} + \sum_{s} \lambda_s c_{s\nu}$$

Among these systems of forces $\mathbf{R}_\nu'$ there exists one, and only one, $\boldsymbol{\rho}_\nu(\rho_{\nu x}, \rho_{\nu y}, \rho_{\nu z})$ such that the vectors $\boldsymbol{\rho}\delta t$ determine some possible displacement of the system $\delta r_\nu (\nu = 1, \ldots, n)$.

Let us note first that if for every possible displacement

$$\sum_\nu \rho_\nu \delta r_\nu = \delta t \sum_\nu \rho_\nu^2 = 0$$

then all $\rho_\nu \equiv$ ($\nu = 1, \ldots, n$). Furthermore, suppose that for every possible displacement

$$\sum_\nu (\rho_{\nu x}\delta x_\nu + \rho_{\nu y}\delta y_\nu + \rho_{\nu z}\delta z_\nu) = \sum_\nu (R_{\nu x}\delta x_\nu + R_{\nu y}\delta y_\nu + R_{\nu z}\delta z_\nu)$$

Substituting into this equation δx_ν, δy_ν, δz_ν from (1.7) we obtain, because of the independence of the δq_i ($i = 1, \ldots, k$), the following k equations:

$$\sum_\nu (\rho_{\nu x}A_{\nu i} + \rho_{\nu y}B_{\nu i} + \rho_{\nu z}C_{\nu i}) = \sum_\nu (R_{\nu x}A_{\nu i} + R_{\nu y}B_{\nu i} + R_{\nu z}C_{\nu i}) \qquad (i = 1, \ldots, k)$$

which we combine with the p equations

$$\sum_\nu (\rho_{\nu x}a_{s\nu} + \rho_{\nu y}b_{s\nu} + \rho_{\nu z}c_{s\nu}) = 0 \qquad (s = 1, \ldots, p)$$

We thus obtain $3n$ linear nonhomogeneous equations for the $3n$ unknowns $\rho_{\nu x}$, $\rho_{\nu y}$, $\rho_{\nu z}$. The determinant of this system of equations is not equal to zero, because in the opposite case the system of the corresponding homogeneous equations would have a nontrivial solution; in other words, there would exist a system of forces $\rho_\nu \not\equiv 0$ for which

$$\sum_\nu \rho_\nu \delta \mathbf{r}_\nu = 0,$$

which, as was pointed out above, is impossible. Hence, there exists one and only one system of quantities ρ_ν($\nu = 1, \ldots, n$) which has the specified properties. On the basis of what has been said it is clear that the force of reaction $\mathbf{R}_\nu$ acting at the point P_ν can be decomposed uniquely into two forces $\mathbf{N}_\nu$ and ρ_ν having the following properties:

1) for every possible displacement δr_ν

$$\sum_\nu \mathbf{N}_\nu \delta \mathbf{r}_\nu = 0 \tag{1.9}$$

2) the vectors $\rho\delta t$ are among the possible displacements, and

$$\sum_\nu \rho_\nu \delta \mathbf{r}_\nu = \tau$$

The force $\mathbf{N}_\nu$ is called the constraint force, the force ρ_ν is the friction force; their projections on the coordinate axes have the following forms:

$$N_{\nu x} = \sum_{s=1}^{p} \lambda_s a_{s\nu}, \qquad N_{\nu y} = \sum_{s=1}^{p} \lambda_s b_{s\nu}, \qquad N_{\nu z} = \sum_{s=1}^{p} \lambda_s c_{s\nu} \tag{1.10}$$

$$\rho_{\nu x} = \sum_{i=1}^{k} \mu_i A_{\nu i}, \qquad \rho_{\nu y} = \sum_{i=1}^{k} \mu_i B_{\nu i}, \qquad \rho_{\nu z} = \sum_{i=1}^{k} \mu_i C_{\nu i}$$

with the same coefficients λ_s and μ_i for all points of the system. We call attention to the fact that the constraint force is, in general, not equal to the normal reaction [1].

If at a given time instant t we know the positions and velocities of the points of the system, and also the given forces $\mathbf{F}_\nu(x_\nu, y_\nu, z_\nu)$, then the constraint forces $\mathbf{N}_\nu$ are determined and will be the same whether there be friction in the system or not.

Indeed, the equations of motion of the points can be written in the form

$$\begin{aligned} m_\nu x_\nu'' &= X_\nu + \sum_s \lambda_s a_{s\nu} + \sum_i \mu_i A_{\nu i} \\ m_\nu y_\nu'' &= Y_\nu + \sum_s \lambda_s b_{s\nu} + \sum_i \mu_i B_{\nu i} \qquad (\nu = 1, \ldots, n) \\ m_\nu z_\nu'' &= Z_\nu + \sum_s \lambda_s c_{s\nu} + \sum_i \mu_i C_{\nu i} \end{aligned} \tag{1.11}$$

Substituting the accelerations x'', y'', z'' given by these equations into the constraint equations (1.3), we obtain a system of p equations which do not contain the forces of friction, and which determine the coefficients λ_s ($s = 1, \ldots, p$) as functions of time, of the coordinates and the velocities of the points, and also of the given forces X_ν, Y_ν, Z_ν.

Therefore, for a frictionless system, a knowledge of the given forces $\mathbf{F}_\nu$ with given initial conditions is completely sufficient for the determination of the motion of the system and of the forces of constraint reactions $\mathbf{N}_\nu$.

For the determination of the motion of a system with friction, it is necessary to know, in addition to the given forces, also the friction forces or, at least, the sum of the elementary work of the reactions for possible displacements.

The friction forces are determined experimentally. Therefore, in the study of the motion of a system with friction one must know, in addition to the given forces, in general also the expressions for the coefficients μ_i in terms of the λ_s, q_i°, $q_j^{\circ\prime}$, t°. The form of these k functions is determined empirically [1].

For the initial conditions q_i°, $q_j^{\circ\prime}$, t° which correspond to static friction and satisfy certain relations of the form

$$h_{\varkappa}(q_j, q_j', t) = 0 \qquad (\varkappa = 1, \ldots, m) \tag{1.12}$$

the functions mentioned are not sufficiently determined. For such initial conditions, the coefficients μ_i are continuous functions of X_ν, Y_ν, Z_ν which may have various forms, depending upon whether or not the functions X_ν, Y_ν, Z_ν satisfy certain inequalities [1] that depend on the particular values q_j°, $q_j^{\circ\prime}$, t° under consideration

One says [1] that the law of friction is known if the experimental data determine the coefficients μ_i in the functions λ_s for arbitrary initial conditions, and in the functions X_ν, Y_ν, Z_ν for special initial conditions satisfying the relations (1.12).

If one has the given expressions for the μ_i $(i = 1, \ldots, k)$, one can replace the λ_s $(s = 1, \ldots, p)$ in them by values obtained by the above-indicated method, and represent the coefficients of μ_i in the form of some functions of time, of the coordinates and velocities of the points of the system, and also of the given forces. It follows that the friction forces will also become known. They will be determined by Formulas (1.10) as functions of the indicated quantities.

Thus, if the law of friction is known, we have $3n$ equations (1.11) for the description of the motion of the system. To these equations we must attach the p constraint equations (1.11), (1.12) and k auxiliary relations obtained from the law of friction.

2. Let us suppose that the law of friction for a given system is known. According to D'Alembert's principle there exists at every instant of time t an equilibrium between the given forces $\mathbf{F}_\nu$, the constraint reactions $\mathbf{R}_\nu = \mathbf{N}_\nu + \rho_\nu$, and the inertia forces $m_\nu \mathbf{w}_\nu$

$$\mathbf{F}_\nu + \mathbf{N}_\nu + \rho_\nu - m_\nu \mathbf{w}_\nu = 0 \qquad (\nu = 1, \ldots, n) \tag{2.1}$$

Here $\mathbf{w}_\nu(x_\nu'', y_\nu'', z_\nu'')$ is the acceleration vector of the point P_ν in the actual motion of the system.

If the system undergoes an arbitrary displacement, then the sum of the elementary work of all forces will be equal to zero. From this and the condition (1.9) we obtain the equation

$$\sum_\nu \{(X_\nu + \rho_{\nu x} - m_\nu x_\nu'')\,\delta x_\nu + (Y_\nu + \rho_{\nu y} - m_\nu y_\nu'')\,\delta y_\nu + \\ + (Z_\nu + \rho_{\nu z} - m_\nu z_\nu'')\,\delta z_\nu\} = 0 \tag{2.2}$$

which states that in the motion of a system with friction the sum of the elementary work of the given forces, the frictional forces, and the forces of inertia for every possible displacement, is equal to zero at any instant of time. Conversely, if for some motion of a constrained system Equation (2.2) is satisfied, then one can deduce Equation (2.1) by adding the equations (2.2) and (1.9), and by making use of the axiom of a free body.

One can, therefore, consider Equation (2.2) as the general equation of dynamics for systems with friction, i.e. as the equation which yields the necessary and sufficient condition for a motion of the system that is compatible with the constraints and the given forces acting under a known law of friction within the given system.

Comparing Equation (2.2) with the Euler-Lagrange principle for a system without friction

$$\sum_{\nu=1}^{n} \{(X_\nu - m_\nu x_\nu'')\,\delta x_\nu + (Y_\nu - m_\nu y_\nu'')\,\delta y_\nu + (Z_\nu - m_\nu z_\nu'')\,\delta z_\nu\} = 0 \tag{2.3}$$

we see that a system with friction can be treated as a frictionless system if one adds to the given forces $\mathbf{F}_\nu(X_\nu, Y_\nu, Z_\nu)$ new active forces which are geometrically equal to the friction forces $\rho_\nu(\rho_{\nu x}, \rho_{\nu y}, \rho_{\nu z})$.

Starting with Equation (2.2), it is not difficult to obtain by the usual method [2] another basic principle of mechanics, the principle of least constraint of Gauss for a system with friction. This principle can be formulated in the following manner.

The motion of a system of material points with friction constraints and subjected to arbitrary forces takes place with the smallest possible constraint [curvature] at any instant of time if one takes as the measure of constraint, applied during an infinitesimal time interval, the sum of the products of the mass of each point by the square of its displacement from the position which it would occupy if it were free and acted upon by the given forces and by forces geometrically equal to the frictional forces.

Let $\gamma_\nu(\gamma_{\nu x}, \gamma_{\nu y}, \gamma_{\nu z})$ be the acceleration vector of the point P_ν of a system that moves in the sense of Gauss, i.e. it has a motion that satisfies the conditions imposed on a constrained system, and the conditions of the constancy of the coordinates x_ν, y_ν, z_ν and the velocities x_ν', y_ν', z_ν' of the system's points for the given instant of time t.

It is easy to see that the vectors $\mathbf{w}_\nu$, γ_ν are among the possible displacements δr_ν of the system [5], i.e. they satisfy

$$\sum_\nu [a_{s\nu}(x_\nu'' - \gamma_{\nu x}) + b_{s\nu}(y_\nu'' - \gamma_{\nu y}) + c_{s\nu}(z_\nu'' - \gamma_{\nu z}) = 0 \qquad (s = 1, \ldots, p) \tag{2.4}$$

According to Gauss's principle, the actual accelerations of the points of a system with friction yield a minimum of the function

$$A = \frac{1}{2}\sum_{\nu} m_{\nu} \left\{\left(\frac{X_{\nu}+\rho_{\nu x}}{m_{\nu}} - x_{\nu}''\right)^2 + \left(\frac{Y_{\nu}+\rho_{\nu y}}{m_{\nu}} - y_{\nu}''\right)^2 + \left(\frac{Z_{\nu}+\rho_{\nu z}}{m_{\nu}} - z_{\nu}''\right)^2\right\} \tag{2.5}$$

and, conversely, the conditions that A be a minimum for accelerations satisfying (1.3) lead to the equations of motion of the system. Indeed, from Equation (2.5) we obtain

$$-\delta A = \sum_{\nu} \{ (X_{\nu} + \rho_{\nu x} - m_{\nu} x_{\nu}'')\, \delta x_{\nu}'' + (Y_{\nu} + \rho_{\nu y} - m_{\nu} y_{\nu}'')\, \delta y_{\nu}'' + $$
$$+ (Z_{\nu} + \rho_{\nu z} - m_{\nu} z_{\nu}'')\, \delta z_{\nu}''\} = 0$$

Here

$$\delta x_{\nu}'' = x_{\nu}'' - \gamma_{\nu x}, \qquad \delta y_{\nu}'' = y_{\nu}'' - \gamma_{\nu y}, \qquad \delta z_{\nu}'' = z_{\nu}'' - \gamma_{\nu z}$$

Multiplying Equation (2.4) by the undetermined factors λ_s, adding the results for s from 1 to p, and combining the sum with the last equation, we obtain the equations of motion of the system

$$m_{\nu} x_{\nu}'' = X_{\nu} + \sum_{s} \lambda_s a_{s\nu} + \rho_{\nu x}, \qquad m_{\nu} y_{\nu}'' = Y_{\nu} + \sum_{s} \lambda_s b_{s\nu} + \rho_{\nu y}$$
$$m_{\nu} z_{\nu}'' = Z_{\nu} + \sum_{s} \lambda_s c_{s\nu} + \rho_{\nu z} \qquad (\nu = 1, \ldots, n)$$

These equations should be augmented by the p constraint equations (1.1) and (1.2), and the k auxiliary relations given earlier.

The equations of motion of a system with friction can also be given in the form of Appell's equations.

Differentiating (1.5) twice with respect to time t, and replacing the q_{l-p_2+1}'', ..., q_l'' on the right-hand sides by their expressions in terms of q_1'', ..., q_k'' obtained from Equations (1.2), we obtain

$$x_{\nu}'' = \sum_{i=1}^{k} A_{\nu i}\, q_i'' + \ldots, \qquad y_{\nu}'' = \sum_{i=1}^{k} B_{\nu i}\, q_i'' + \ldots$$
$$z_{\nu}'' = \sum_{i=1}^{k} C_{\nu i}\, q_i'' + \ldots$$

Here, and in the sequel, the dots indicate independent terms that do not involve q_1'', ..., q_k''.

Making use of the last formulas, we can express (2.5) in the form

$$A = S(q_1'', \ldots, q_k'') - \sum_{i=1}^{k} Q_i\, q_i'' - \sum_{i=1}^{k} \Phi_i\, q_i'' + \ldots$$

where S denotes the energy of acceleration

$$S = \frac{1}{2}\sum_{\nu} m_{\nu} (x_{\nu}''^2 + y_{\nu}''^2 + z_{\nu}''^2)$$

given in the indicated way as a function of q_1'', ..., q_k'' and

$$Q_i = \sum_{\nu} (X_{\nu}A_{\nu i} + Y_{\nu}B_{\nu i} + Z_{\nu}C_{\nu i})$$

$$\Phi_i = \sum_{\nu} (\rho_{\nu x}A_{\nu i} + \rho_{\nu y}B_{\nu i} + \rho_{\nu z}C_{\nu i}) \qquad (i = 1,\ldots,k)$$

denote generalized given forces and frictional forces.

From the condition that the function A attain its minimum for the accelerations q_1'', ..., q_k'', we obtain Appell's equation

$$\frac{\partial S}{\partial q_i''} = Q_i + \Phi_i \qquad (i = 1, \ldots, k)$$

for a system with friction.

3. Expression (2.5) for the deviation of A contains explicitly the frictional forces, i.e. the components of the constraint forces. However, what is more interesting is the establishment of Gauss's principle for a system with friction without the explicit appearance of the frictional forces in the constraint function.

The possible displacements of a given system with friction are determined by Equations (1.4). In many cases it is possible to select from the family of possible displacements certain ones for which the frictional forces perform no work.

Indeed, adhering to the hypothesis that we know the law of friction, let us assume that among the possible displacements there is one which satisfies the conditions

$$\rho_{\nu x}\delta x_{\nu} + \rho_{\nu y}\delta y_{\nu} + \rho_{\nu z}\delta z_{\nu} = 0 \qquad (\nu = 1, \ldots, n) \tag{3.1}$$

For the existence of such not identically vanishing displacements it is necessary and sufficient that the rank of the matrix of the coefficients of the $p + n$ equations (1.4) and (3.1) be less than the number $3n$ of the variations δx_{ν}, δy_{ν}, δz_{ν} ($\nu = 1, \ldots, n$), or, what is the same thing, that $k > n$.

It is obvious that for these displacements

$$\sum_{\nu} (R_{\nu x}\delta x_{\nu} + R_{\nu y}\delta y_{\nu} + R_{\nu z}\delta z_{\nu}) = 0 \tag{3.2}$$

The set of the possible displacements, satisfying the conditions (3.1) will be called, for the sake of brevity, (c)-displacements [3].

For arbitrary (c)-displacements Equation (2.2) takes on the form

$$\sum_{\nu} \{(X_\nu - m_\nu x_\nu'')\,\delta x_\nu + (Y_\nu - m_\nu y_\nu'')\,\delta y_\nu + (Z_\nu - m_\nu z_\nu'')\,\delta z_\nu = 0 \qquad (3.3)$$

The constraint reactions do not enter into this equation, and it plays the role of a principle for a system with friction, analogous to the Euler-Lagrange principle.

Let us now proceed to the establishment of a principle of Gauss's type by starting with Equation (3.3). Among the acceleration vectors γ_ν of the point P_ν we select vectors γ_ν^c such that the vectors $w_\nu - \gamma_\nu^c$ are among the (c)-displacements, i.e. satisfy the conditions

$$\rho_{\nu x}\delta x_\nu'' + \rho_{\nu y}\delta y_\nu'' + \rho_{\nu z}\delta z_\nu'' = 0 \qquad (\nu = 1, \ldots, n) \qquad (3.4)$$

where

$$\delta x_\nu'' = x_\nu'' - \gamma_{\nu x}^c, \qquad \delta y_\nu'' = y_\nu'' - \gamma_{\nu y}^c, \qquad \delta z_\nu'' = z_\nu'' - \gamma_{\nu z}^c$$

For such possible [virtual] (c)-motions Equation (3.3) takes on the form

$$\sum_\nu m_\nu \left\{\left(x_\nu'' - \frac{X_\nu}{m_\nu}\right)(x_\nu'' - \gamma_{\nu x}^c) + \left(y_\nu'' - \frac{Y_\nu}{m_\nu}\right) \times \right.$$
$$\left. \times\, (y_\nu'' - \gamma_{\nu y}^c) + \left(z_\nu'' - \frac{Z_\nu}{m_\nu}\right)(z_\nu'' - \gamma_{\nu z}^c) \right\} = 0$$

which can easily be expressed in the form

$$A_{w\gamma^c} + A_{wv} - A_{v\gamma^c} = 0$$

where

$$A_{wv} = \frac{1}{2}\sum_\nu m_\nu \left[\left(x_\nu'' - \frac{X_\nu}{m_\nu}\right)^2 + \left(y_\nu'' - \frac{Y_\nu}{m_\nu}\right)^2 + \left(z_\nu'' - \frac{Z_\nu}{m_\nu}\right)^2\right]$$

denotes the degree of deviation (the constraint) of the actual motion of the system with friction from the actual motion of the system freed from all connections: in an analogous manner one can determine $A_{w\gamma^c}$ and $A_{v\gamma^c}$.

From this equation we obtain two inequalities

$$A_{w\gamma^c} < A_{v\gamma^c}, \qquad A_{wv} < A_{v\gamma^c} \qquad (3.5)$$

Thus, we have proved the following theorems.

1. The deviation of the actual motion (w_ν) of the system with friction from a possible (c)-motion is less than the deviation of the latter from the motion of the system freed from all connections.

2. The deviation of the actual motion of the system with friction freed from all constraints is less than the deviation of the latter from the possible [virtual] (c)-motion.

It follows from this that at every instant of time the actual accelerations of the points of a system with friction cause the first variation of the constraint

$$A_1 = \frac{1}{2}\sum_\nu m_\nu \left\{\left(\frac{X_\nu}{m_\nu} - x_\nu''\right)^2 + \left(\frac{Y_\nu}{m_\nu} - y_\nu''\right)^2 + \left(\frac{Z_\nu}{m_\nu} - z_\nu''\right)^2\right]$$

to vanish if $\delta x_\nu''$, $\delta y_\nu''$, $\delta z_\nu''$ $(\nu = 1, \ldots, n)$ satisfy the conditions (2.4) and (3.4).

In other words, Gauss's principle for a system with friction can be formulated in the same way as for a frictionless system if one takes into consideration only possible (c)-motions.

The equations of motion of a system with friction can be derived from Gauss's principle in the new form in the usual manner [2].

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